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Role of non-resonant interactions in the evolution of nonlinear random water wave fields

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We present the results of direct numerical simulations (DNS) of the evolution of nonlinear random water wave fields. The aim of the work is to validate the hypotheses underlying the statistical theory of nonlinear dispersive waves and to clarify the role of exactly resonant, nearly resonant and non-resonant wave interactions. These basic questions are addressed by examining relatively simple wave systems consisting of a finite number of wave packets localized in Fourier space. For simulation of the long-term evolution of random water wave fields we employ an efficient DNS approach based on the integrodifferential Zakharov equation. The non-resonant cubic terms in the Hamiltonian are excluded by the canonical transformation. The proposed approach does not use a regular grid of harmonics in Fourier space. Instead, wave packets are represented by clusters of discrete Fourier harmonics.

The simulations demonstrate the key importance of near-resonant interactions for the nonlinear evolution of statistical characteristics of wave fields, and show that simulations taking account of only exactly resonant interactions lead to physically meaningless results. Moreover, exact resonances can be excluded without a noticeable effect on the field evolution, provided that near-resonant interactions are retained. The field evolution is shown to be robust with respect to the details of the account taken of near-resonant interactions. For a wave system initially far from equilibrium, or driven out of equilibrium by an abrupt change of external forcing, the evolution occurs on the 'dynamical' time scale, that is with quadratic dependence on nonlinearity ε, not on the $O(\varepsilon^{-4})$ time scale predicted by the standard statistical theory. However, if a wave system is initially close to equilibrium and evolves slowly in the presence of an appropriate forcing, this evolution is in quantitative accordance with the predictions of the kinetic equation. We suggest a modified version of the kinetic equation able to describe all stages of evolution.

Although the dynamic time scale of quintet interactions ε^{-3} is smaller than the kinetic time scale ε^{-4}, they are not included in the existing statistical theory, and their effect on the evolution of wave spectra is unknown. We show that these interactions can significantly affect the spectrum evolution, although on a time scale much larger than $O(\varepsilon^{-4})$. However, for waves of high but still realistic steepness $\varepsilon \sim 0.25$, the scales of evolution are no longer separated. By tracing the evolution of high statistical moments of the wave field, we directly verify one of the main assumptions used in the derivation of the kinetic equation: the quasi-Gaussianity of the wave holds throughout the evolution, both with and without accounting for quintet interactions.

The conclusions are not confined to water waves and are applicable to a generic weakly nonlinear dispersive wave field with prohibited triad interactions.

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1. Introduction

The adequate statistical description of 'wave turbulence', or more precisely, the nonlinear evolution of random wave fields is one of the open fundamental problems of fluid mechanics, and of physics in general, although the advances in understanding over last few decades are quite considerable. The main advances are primarily confined to weakly nonlinear strongly dispersive wave fields with an infinite number of degrees of freedom, for which a closed equation for the evolution of the second-order statistical moments has been derived, extensively studied and successfully applied to wave turbulence of various physical origins from liquid helium to interstellar plasmas (e.g. Zakharov, L'vov & Falkovich 1992). In the water wave context, this equation was first derived by Hasselmann (1962). This equation is known in different branches of science under different names as the spectral transfer, the Hasselmann, the Boltzmann, or the kinetic equation (KE); we will use the latter term. In the present paper we, with minimal loss of generality (which we discuss later), will confine our consideration of the evolution of random wave fields to the water wave context, due to the fundamental nature of this problem and to its central role in the important practical task of wind wave forecasting.

The approach to the statistical description of the nonlinear evolution of random wave fields resulting in the derivation of the KE exploits weak nonlinearity of the wave fields and is based upon a number of hypotheses, which vary depending on the specific derivation employed and will be briefly discussed below. It is important to note that a next order approximation for the KE does not exist. Although this fact is not exceptional for asymptotic results, since the range of validity of the equation has not been established, the theory needs an independent corroboration.

The most obvious way to check both the assumptions underlying the derivation of the KE and its range of validity is the direct numerical simulation (DNS) of statistical ensembles of random wave fields. Although this task appears to be straightforward, there are fundamental difficulties in applying DNS to this problem. Here, we will mention just two of them. First, the hydrodynamic equations should be integrated for a very large number of modes over quite large time intervals, much larger than those required in simulations of classical turbulence. Since, by the definition of wave turbulence, nonlinearity of the wave field is small, this smallness being characterized by a small parameter, say, ε, the necessary times are at least $O(\varepsilon^{-4})$ of characteristic wave periods for the media with prevailing quartet interaction, as is the case for water waves. In such simulations the 'fast' time dependence cannot be filtered out, and very high accuracy of the simulations is required. Second, the KE describes continuous wave fields and the δ-functions in frequency in the integrand of the KE are the result of a limit process essentially based on the existence of a continuum of waves involved in both resonant and non-resonant interactions. In this context it is not *a priori* clear how to perform the unavoidable discretization of a continuous wave field.

The importance of the open problem of the verification and generalization of the KE was realized quite early, and the idea that the DNS is the most promising way to tackle it is well-established. Although there exists a large variety of algorithms developed to simulate the evolution of water waves, none of them has proved to be particularly well suited to the challenge (for the reasons discussed below). Nevertheless, many different groups have attacked this problem by employing some modifications of the existing algorithms, although with a limited success (Onorato *et al.* 2002; Tanaka 2001*a*, *b*; Willemsen 2001; Dyachenko, Korotkevich & Zakharov 2004; Yokoyama 2004). These papers were much more modestly aimed at a verification that the KE predictions are at least very roughly consistent with the DNS. The most robust predictions of

the kinetic equation for all dispersive waves with prohibited triad interactions are: (i) the existence of a Kolmogorov-type energy cascade towards small scales – the 'direct cascade', which in terms of energy frequency spectra $E(\omega)$ manifests itself as a power-like spectrum (in the water wave context $E(\omega) \sim \omega^{-4}$); (ii) the scaling of energy fluxes and evolution times as $\sim \varepsilon^6$ and $\sim \varepsilon^{-4}$ respectively. Based upon coarse mesh simulations, a preliminary observation of the existence of the direct cascade for water waves was reported by Willemsen (2001). Tanaka (2001 *a*, *b*), having simulated just the very initial stage of wave field evolution (25 wave periods), found for typical water wave spectra an agreement in energy fluxes between the DNS and KE, and confirmed the ε^6 scaling. In the recent works by Onorato *et al.* (2002), Dyachenko *et al.* (2004) and Yokoyama (2004) the field evolution was simulated over sufficiently large time scales ($O(10^3)$ characteristic wave periods). It was demonstrated that there is indeed a direct cascade resulting in the formation of stationary power-like spectra that are in a reasonable agreement with the predictions of the KE. However, the same ω^{-4} stationary spectra could be also obtained from scaling-type arguments even without invoking the concept of resonant interactions (Kitaigorodskii 1962) and therefore, in itself, the finding of ω^{-4} spectra is not the decisive argument.

Thus, the issue of quantitative verification of the KE remained open. It also became apparent that a specially designed numerical tool is needed. This led to a new set of basic questions stemming from the necessity to discretize the wave field while preserving the desired properties of continuous equations. The questions include:

(i) Is it possible in principle to develop an efficient numerical scheme employing a discretization in Fourier space for an accurate simulation of the evolution of statistical characteristics of wave ensembles, retaining the fundamental properties of continuous wave fields?

(ii) What is the role of resonant, approximately resonant, and non-resonant interactions?

(iii) What are the actual time scales of field evolution and their relation to the KE characteristic evolution times?

(iv) What is the role of higher-order resonant interactions and what is required to take them into account in DNS?

The present paper addresses this challenge. We put the strongest emphasis on the control of accuracy and transparency of the simulations. To this end we focus upon a 'laboratory type' setting of simulations; that is, we investigate in detail the behaviour of toy models of the wave field made up of a finite number of spectrally narrow wave packets that are in resonance. In parallel, we clarify the answers to questions raised in further refining the algorithm developed earlier for this type of simulation based upon the Zakharov equation (Annenkov & Shrira 2001).

It is important to note that the KE is not only based on the key assumption of quasi-Gaussianity, but in fact represents a large-time limit of the theory. In this limit, the evolution time scales are proportional to ε^{-4} and only exactly resonant interactions are believed to be important. Implicitly it is also presumed that the evolution that occurs at shorter time scales, before the assumptions used in the derivation of the KE become valid, is insignificant. However, this has never been tested. Simulations of the short-term evolution of capillary wave spectra, performed by Watson & Buchsbaum (1996) and Watson (1999), showed that taking account of interactions that are not exactly resonant leads to significant corrections to the energy transfer in the capillary wave field. We demonstrate the crucial role of near-resonant interactions at all stages of evolution, and also show that due to near-resonant interactions a considerable part of the field evolution does occur before the KE 'turns

on'. This can lead either to effective modifications of the actual initial conditions for the KE, or, more importantly, to an evolution that is significantly different from the KE predictions.

The paper is organized as follows. In § 2 we begin with the basic equations and a very brief discussion of assumptions leading to the KE, which are to be checked in other sections. For the DNS, we use the algorithm for the numerical integration of the Zakharov equation, previously derived by the authors and successfully used in a number of physical problems related to surface wave evolution. The basics of this algorithm are also briefly outlined.

In § 3, we choose, as the first meaningful example, one of the simplest wave systems with non-trivial evolution of statistical characteristics: the single exactly resonant quartet. It is demonstrated that in order to obtain, with DNS, an evolution that is at least qualitatively similar to the KE solution for this system, it is essential to take into account the neighbourhood of the exact resonances, i.e. to supplement the resonant quartet with approximately resonant interactions. This is done, in a systematic and robust (that is, without a significant dependence on any extra parameters) way by introducing the concept of 'clusters'. In physical terms, this corresponds to the idea of considering numerically the interaction of wave packets, instead of the interaction of discrete harmonics. It is demonstrated that this DNS approach is able to model quantitatively the evolution of systems consisting of any number of resonantly interacting wave packets.

In § 4, we perform a quantitative comparison of the DNS and KE simulations. While the KE solutions have time scale proportional to ε^{-4} uniformly in time, the results of DNS demonstrate ε^{-2} scaling at the initial stages of the evolution, which usually means much faster evolution. This leads to important consequences for all wave systems, especially for those adjusting to rapidly varying external forcing.

The role of the next-order (five-wave) nonlinear interactions, for which no analytical theory exists, is studied in § 5. We show that these effects lead to a non-trivial evolution of the spectra for large time scales. Moreover, for moderately (but realistically) steep waves, these time scales are found to be not well separated from the four-wave interaction time scale, and the evolution of a wave system can be significantly affected by taking account of higher-order processes. Concluding remarks and a brief discussion are in § 6.

2. The statement of the problem and the numerical method

In this section we briefly review the derivation of the Zakharov equation, which is the starting point of our numerical analysis, the numerical algorithm itself, and the assumptions usually made in the derivation of the KE which we will test in subsequent sections. As we mentioned earlier, we have chosen water waves as the most representative example of a wave field where the description of random field evolution is important. Following the established paradigm, we consider potential gravity waves on the free surface of a homogeneous, incompressible and inviscid fluid of infinite depth and transform the original equations into an integrodifferential equation, in terms of new appropriately chosen variables (Zakharov 1968). The resulting equation is usually referred to as the Zakharov equation.

2.1. *The Zakharov equation*

We choose a coordinate system with the origin located at the undisturbed water surface, the vertical axis z oriented upward and the horizontal axes x, y. Let

$z=\zeta(\boldsymbol{x},t)$ specify the free surface, and let $\varphi(\boldsymbol{x},z,t)$ be the velocity potential, with $\psi(\boldsymbol{x},t)=\varphi(\boldsymbol{x},\zeta(\boldsymbol{x},t),t)$ being the potential at the surface. Then the governing equations can be written in the Hamiltonian form

$$\frac{\partial\zeta(\boldsymbol{x},t)}{\partial t}=\frac{\delta H}{\delta\psi(\boldsymbol{x},t)},\qquad \frac{\partial\psi(\boldsymbol{x},t)}{\partial t}=-\frac{\delta H}{\delta\zeta(\boldsymbol{x},t)}, \tag{2.1a, b}$$

where δ denotes the operator of functional differentiation, and the Hamiltonian H is the total energy of the system, namely

$$H=\frac{1}{2}\int_{-\infty}^{\zeta}\left[(\nabla\varphi)^2+\left(\frac{\partial\varphi}{\partial z}\right)^2\right]\mathrm{d}z\,\mathrm{d}\boldsymbol{x}+\frac{1}{2}g\int\zeta^2\,\mathrm{d}\boldsymbol{x}, \tag{2.2}$$

where integration with respect to $\boldsymbol{x}$ over the entire horizontal plane is implied. Here and below, we retain only one integral sign in multiple integrations.

In order to obtain a closed system in ψ, ζ, one has to calculate H in terms of these variables. An approximate solution can be obtained, by performing an expansion in powers of wave steepness. Making the Fourier transformation

$$\left.\begin{aligned}\zeta(\boldsymbol{x})&=\frac{1}{2\pi}\int\zeta(\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k},\\ \psi(\boldsymbol{x})&=\frac{1}{2\pi}\int\psi(\boldsymbol{k})\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{d}\boldsymbol{k},\end{aligned}\right\} \tag{2.3}$$

where integration is performed over the entire $\boldsymbol{k}$-plane, $\boldsymbol{k}=(k_x,k_y)$, and introducing complex variables $a(\boldsymbol{k})$,

$$\zeta(\boldsymbol{k})=M(\boldsymbol{k})[a(\boldsymbol{k})+a^*(-\boldsymbol{k})],\qquad \psi(\boldsymbol{k})=-\mathrm{i}N(\boldsymbol{k})[a(\boldsymbol{k})-a^*(-\boldsymbol{k})], \tag{2.4}$$

where

$$M(\boldsymbol{k})=\left[\frac{q(\boldsymbol{k})}{2\omega(\boldsymbol{k})}\right]^{1/2},\qquad N(\boldsymbol{k})=\left[\frac{\omega(\boldsymbol{k})}{2q(\boldsymbol{k})}\right]^{1/2},$$

equations (2.1a, b) take the form

$$\mathrm{i}\frac{\partial a(\boldsymbol{k})}{\partial t}=\frac{\delta H}{\delta a^*(\boldsymbol{k})}, \tag{2.5}$$

where asterisk means complex conjugation, $\omega(\boldsymbol{k})=[gq(\boldsymbol{k})]^{1/2}$ is the linear dispersion relation, $q(\boldsymbol{k})=|\boldsymbol{k}|=k$ for infinite depth. Without the loss of generality, $g=1$, with the corresponding change of length scale. In (2.5), the Hamiltonian H is a functional of $a(\boldsymbol{k})$, $a^*(\boldsymbol{k})$, and, in the form of a series in powers of these variables, in the generic case can be written as

$$\begin{aligned}H=&\int\omega_0a_0^*a_0\,\mathrm{d}\boldsymbol{k}_0\\ &+\int U^{(1)}_{012}(a_0^*a_1a_2+a_0a_1^*a_2^*)\delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{012}\\ &+\frac{1}{3}\int U^{(3)}_{012}(a_0^*a_1^*a_2^*+a_0a_1a_2)\delta_{0+1+2}\,\mathrm{d}\boldsymbol{k}_{012}\\ &+\int V^{(1)}_{0123}(a_0^*a_1a_2a_3+a_0a_1^*a_2^*a_3^*)\delta_{0-1-2-3}\,\mathrm{d}\boldsymbol{k}_{0123}\\ &+\frac{1}{2}\int V^{(2)}_{0123}a_0^*a_1^*a_2a_3\delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{0123}\end{aligned}$$

$$+\frac{1}{4}\int V^{(4)}_{0123}(a_0^* a_1^* a_2^* a_3^* + a_0 a_1 a_2 a_3)\delta_{0+1+2+3}\,\mathrm{d}\boldsymbol{k}_{0123}$$
$$+\int W^{(1)}_{01234}(a_0^* a_1 a_2 a_3 a_4 + a_0 a_1^* a_2^* a_3^* a_4^*)\delta_{0-1-2-3-4}\,\mathrm{d}\boldsymbol{k}_{01234}$$
$$+\frac{1}{2}\int W^{(2)}_{01234}(a_0^* a_1^* a_2 a_3 a_4 + a_0 a_1 a_2^* a_3^* a_4^*)\delta_{0+1-2-3-4}\,\mathrm{d}\boldsymbol{k}_{01234}$$
$$+\frac{1}{5}\int W^{(5)}_{01234}(a_0^* a_1^* a_2^* a_3^* a_4^* + a_0 a_1 a_2 a_3 a_4)\delta_{0+1+2+3+4}\,\mathrm{d}\boldsymbol{k}_{01234}$$
$$+\cdots. \tag{2.6}$$

The real interaction coefficients $U^{(n)}$, $V^{(n)}$, $W^{(n)}$ are known functions of the wavenumbers $\boldsymbol{k}_j$ and frequencies ω_j, given in Krasitskii (1994). We have used here compact notation, replacing the arguments $\boldsymbol{k}_j$ of all the corresponding functions by the subscripts j, assigning the subscript zero to $\boldsymbol{k}$. Thus, $a_j = a(\boldsymbol{k}_j, t)$, $\omega_j = \omega(\boldsymbol{k}_j)$, $U^{(n)}_{012} = U^{(n)}(\boldsymbol{k}_0, \boldsymbol{k}_1, \boldsymbol{k}_2)$, $\delta_{0-1-2} = \delta(\boldsymbol{k}_0 - \boldsymbol{k}_1 - \boldsymbol{k}_2)$, where δ is the Dirac delta function, etc. In the same way, $\mathrm{d}k_0 = \mathrm{d}k$, $\mathrm{d}k_{012} = \mathrm{d}k_0\mathrm{d}k_1\mathrm{d}k_2$, etc, and the integration is again performed over the entire $\boldsymbol{k}$-plane.

It is well-known (Zakharov 1968; Krasitskii 1994) that the numerous terms in the expansion (2.6) do not have equal significance; at each order m, only the processes satisfying the resonant conditions

$$\sum_{j=1}^{m} s_j \boldsymbol{k}_j = 0, \qquad \sum_{j=1}^{m} s_j \omega_j = 0, \tag{2.7}$$

where $s_j = \pm 1$, are essential for the dynamics. If some combinations of signs s_j are prohibited by the dispersion relation, the corresponding terms can be removed from expansion (2.6) by a special canonical transformation, allowing one to obtain the so-called 'effective Hamiltonian' (Zakharov 1968). For the case of gravity waves, considering the expansion up to ε^5, only two resonant processes are permitted:

$$k_1 + k_2 - k_3 - k_4 = 0, \qquad \omega_1 + \omega_2 - \omega_3 - \omega_4 = 0, \tag{2.8}$$

and

$$k_1 + k_2 - k_3 - k_4 - k_5 = 0, \qquad \omega_1 + \omega_2 - \omega_3 - \omega_4 - \omega_5 = 0. \tag{2.9}$$

Consider a canonical transformation from $a(\boldsymbol{k})$ to a new variable $b(\boldsymbol{k})$, postulating it in the form of integral-power series (Zakharov 1968; Krasitskii 1994):

$$a_0 = b_0 + \int A^{(1)}_{012} b_1 b_2 \delta_{0-1-2}\,\mathrm{d}\boldsymbol{k}_{12}$$
$$+\int A^{(2)}_{012} b_1^* b_2 \delta_{0+1-2}\,\mathrm{d}\boldsymbol{k}_{12} + \int A^{(3)}_{012} b_1^* b_2^* \delta_{0+1+2}\,\mathrm{d}\boldsymbol{k}_{12}$$
$$+\int B^{(1)}_{0123} b_1 b_2 b_3 \delta_{0-1-2-3}\,\mathrm{d}\boldsymbol{k}_{123} + \int B^{(2)}_{0123} b_1^* b_2 b_3 \delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{123}$$
$$+\int B^{(3)}_{0123} b_1^* b_2^* b_3 \delta_{0+1+2-3}\,\mathrm{d}\boldsymbol{k}_{123} + \int B^{(4)}_{0123} b_1^* b_2^* b_3^* \delta_{0+1+2+3}\,\mathrm{d}\boldsymbol{k}_{123}$$
$$+\int C^{(1)}_{01234} b_1 b_2 b_3 b_4 \delta_{0-1-2-3-4}\,\mathrm{d}\boldsymbol{k}_{1234} + \int C^{(2)}_{01234} b_1^* b_2 b_3 b_4 \delta_{0+1-2-3-4}\,\mathrm{d}\boldsymbol{k}_{1234}$$

$$+\int C^{(3)}_{01234} b_1^* b_2^* b_3 b_4 \delta_{0+1+2-3-4}\,\mathrm{d}\boldsymbol{k}_{1234} + \int C^{(4)}_{01234} b_1^* b_2^* b_3^* b_4 \delta_{0+1+2+3-4}\,\mathrm{d}\boldsymbol{k}_{1234}$$
$$+\int C^{(5)}_{01234} b_1^* b_2^* b_3^* b_4^* \delta_{0+1+2+3+4}\,\mathrm{d}\boldsymbol{k}_{1234} + \cdots. \tag{2.10}$$

The Hamiltonian system (2.5) under the transformation (2.10) has the form

$$\mathrm{i}\frac{\partial b(\boldsymbol{k})}{\partial t} = \frac{\delta \tilde{H}}{\delta b^*(\boldsymbol{k})}, \tag{2.11}$$

where $\tilde{H} = \tilde{H}(b, b^*)$. The specific choice of the coefficients $A^{(n)}$, $B^{(n)}$, $C^{(n)}$ enables reduction of the Hamiltonian, in other words, an important simplification of $\tilde{H}$, which now contains resonant terms only. For the case of purely gravity waves, the reduced Hamiltonian has the form

$$\tilde{H} = \int \omega_0 b_0 b_0^*\,\mathrm{d}\boldsymbol{k}_0 + \frac{1}{2}\int T_{0123} b_0^* b_1^* b_2 b_3 \delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{123}$$
$$+\frac{1}{2}\int \tilde{W}^{(2)}_{01234}(b_0^* b_1^* b_2 b_3 b_4 + b_0 b_1 b_2^* b_3^* b_4^*)\delta_{0+1-2-3-4}\,\mathrm{d}\boldsymbol{k}_{1234} + \cdots, \tag{2.12}$$

so that the corresponding reduced equation, to the fifth order in ε, is

$$\mathrm{i}\frac{\partial b_0}{\partial t} = \omega_0 b_0 + \int T_{0123} b_1^* b_2 b_3 \delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{123}$$
$$+\int \tilde{W}^{(2)}_{01234} b_1^* b_2 b_3 b_4 \delta_{0+1-2-3-4}\,\mathrm{d}\boldsymbol{k}_{1234}$$
$$+\frac{3}{2}\int \tilde{W}^{(2)}_{43210} b_1^* b_2^* b_3 b_4 \delta_{0+1+2-3-4}\,\mathrm{d}\boldsymbol{k}_{1234}. \tag{2.13}$$

Equation (2.13), being a generalization of the original result of Zakharov (1968) to the next order in ε, is known as the five-wave Zakharov equation for gravity waves. The interaction coefficients of the reduced equation, T and $\tilde{W}^{(2)}$, are known in terms of $U^{(n)}$, $V^{(2)}$ and $U^{(n)}$, $V^{(n)}$, $W^{(2)}$ respectively. The corresponding expressions can be found in Krasitskii (1994). This equation is quite general; it could be derived for almost any weakly nonlinear dispersive wave which is primarily conservative and where triad interactions are prohibited. The specifics of each type of wave are accumulated in the expressions for the coefficients.

2.2. *Numerical algorithm*

It is convenient to eliminate the rapid change of phase by change of variable:

$$b(\boldsymbol{k}, t) = B(\boldsymbol{k}, t)\exp[-\mathrm{i}\omega(\boldsymbol{k})t],$$

obtaining

$$\mathrm{i}\frac{\partial B_0}{\partial t} = \int T_{0123} B_1^* B_2 B_3 \mathrm{e}^{\mathrm{i}(\omega_0+\omega_1-\omega_2-\omega_3)t}\delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{123}$$
$$+\int \tilde{W}^{(2)}_{01234} B_1^* B_2 B_3 B_4 \mathrm{e}^{\mathrm{i}(\omega_0+\omega_1-\omega_2-\omega_3-\omega_4)t}\delta_{0+1-2-3-4}\,\mathrm{d}\boldsymbol{k}_{1234}$$
$$+\frac{3}{2}\int \tilde{W}^{(2)}_{43210} B_1^* B_2^* B_3 B_4 \mathrm{e}^{\mathrm{i}(\omega_0+\omega_1+\omega_2-\omega_3-\omega_4)t}\delta_{0+1+2-3-4}\,\mathrm{d}\boldsymbol{k}_{1234}. \tag{2.14}$$

A method for the numerical integration of the Zakharov equation was proposed in Annenkov & Shrira (2001). The essence of the approach is the efficient computational

strategy, where all the coefficients T and $\tilde{W}$ are computed by a preprocessing routine and stored in a way that facilitates all subsequent operations of integration in time. The algorithm was successfully applied to integrate dynamic equations in a number of problems (Annenkov & Shrira 2001).

It is important to note that the variables $b(\boldsymbol{k})$ and $B(\boldsymbol{k})$ are not physical wave amplitudes, but appear as a result of the canonical transformation (2.10). In the linear approximation, $b(\boldsymbol{k}) = a(\boldsymbol{k})$; but at the first nonlinear order, $O(\varepsilon)$ terms due to non-resonant three-wave interactions appear and must be taken into account when transforming the solutions of the Zakharov equation back to physical space (e.g. Creamer *et al.* 1989). However, we do not perform this transformation to physical space, since the main objective of the present paper is the comparison of the DNS results with the predictions of the statistical theory for water waves, which are formulated in the same canonically transformed space.

2.3. *Statistical description*

Now, let us turn to statistical description of a wave system in terms of the correlation functions of the field, $b(\boldsymbol{k}, t)$. The classical derivation procedure, described in detail in, e.g., Zakharov *et al.* (1992), uses (2.13), with five-wave terms dropped, as the starting point and leads to the equation

$$\frac{\partial n_0}{\partial t} = 4\pi \int T_{0123}^2 f_{0123} \delta_{0+1-2-3} \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3)\, \mathrm{d}\boldsymbol{k}_{123}. \tag{2.15}$$

Here, n_0 is the second-order correlator,

$$\langle b_0^* b_1 \rangle = n_0 \delta_{0-1}, \tag{2.16}$$

where angular brackets mean ensemble averaging, and

$$f_{0123} = n_2 n_3 (n_0 + n_1) - n_0 n_1 (n_2 + n_3). \tag{2.17}$$

Equation (2.15) is essentially of the type of the Boltzmann equation for the evolution of a statistical ensemble, although in the context of surface water waves it is often referred to as the Hasselmann or the 'spectral transfer' equation; in this article, as we mentioned in the Introduction, we prefer to use the term 'kinetic' equation to emphasize its universality and relevance for all dispersive wave systems with prohibited triads. In its derivation from (2.13), a number of hypotheses and assumptions have been used (Zakharov *et al.* 1992). We will now briefly review the most common way of deriving (2.15), pointing out the approximations involved.

The wave field is taken to be statistically homogeneous in space, as implied by the form of the correlator (2.16). Then, the free-wave field, obtained at the zeroth approximation in ε, is assumed to have Gaussian statistics, for which all odd-order correlators vanish, and the fourth-order correlator decomposes into products of pair correlators,

$$J_{0123}^{(0)} \delta_{0+1-2-3} = \langle b_0^* b_1^* b_2 b_3 \rangle = n_0 n_1 (\delta_{0-2}\delta_{1-3} + \delta_{0-3}\delta_{1-2}). \tag{2.18}$$

At the next approximation, correlations due to resonant nonlinear interactions result in the fourth-order cumulant $J_{0123}^{(1)}$. The essential hypothesis is that wave field is presumed to remain quasi-Gaussian over the time scale of evolution; therefore, for all times the cumulant should remain small compared to $J_{0123}^{(0)}$.

Using the Zakharov equation (2.13), with the five-wave interaction terms dropped, we calculate $\partial n_0/\partial t$:

$$\frac{\partial n_0}{\partial t} = 2\mathrm{Im}\int T_{0123} J^{(1)}_{0123}\delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{123}. \tag{2.19}$$

The cumulant $J^{(1)}_{0123}$ is specified by an evolution equation containing on the right-hand side the sixth-order correlator I_{012345}, which by invoking the quasi-Gaussianity assumption is replaced by the corresponding free-field Gaussian correlator $I^{(0)}_{012345}$ representable in terms of the products of pair correlators. As a result we have

$$\left(\mathrm{i}\frac{\partial}{\partial t} + \Delta\omega\right) J^{(1)}_{0123} = 2T_{0123} f_{0123}, \tag{2.20}$$

where $\Delta\omega = \omega_0 + \omega_1 - \omega_2 - \omega_3$, and f_{0123} is specified by (2.17). It is usually assumed that n_0 and, hence, f_{0123} depends on slow time μt, such that $\mu/\Delta\omega \ll 1$. Then the closed equation for the evolution of $n(t)$ in its classical form (2.15) is obtained (see e.g. Zakharov *et al.* 1992) by making the following steps. Employing the assumed wide separation of scales and neglecting the oscillating terms $\sim \mathrm{e}^{-\mathrm{i}\Delta\omega t}$ in the full solution to (2.20), an approximate solution for large t that depends only on the slow time scale,

$$J^{(1)}_{0123}(t) = \frac{2T_{0123}}{\Delta\omega} f_{0123}, \tag{2.21}$$

is substituted into (2.19). Here this solution is understood in terms of generalized functions:

$$J^{(1)}_{0123}(t) = 2T_{0123}\left[\frac{P}{\Delta\omega} + \mathrm{i}\pi\delta(\Delta\omega)\right] f_{0123}(t), \tag{2.22}$$

where P stands for 'principal value'. Note that is commonly assumed that $\mu \sim \varepsilon^4$ while $\Delta\omega$ for four-wave interactions is $O(\varepsilon^2)$. Thus, $\mu/\Delta\omega \sim \varepsilon^2 \ll 1$, and the asymptotic derivation is valid as long as our interest is confined to slow $O(\varepsilon^{-4})$ evolution.

Alternative derivations of the same kinetic equation could be based upon a somewhat different set of hypotheses (e.g. Benney & Saffman 1966; Reznik 1983), assuming, instead of quasi-Gaussianity, a certain smoothness of the cumulants in the $\boldsymbol{k}$-space at the initial moment. This smoothness is understood in terms of slow time and, in this sense, the initial stage of evolution is also ignored.

Thus, the derivation of the kinetic equation is based on a number of important hypotheses that are difficult to verify from within the approach, and an independent verification by DNS is needed. Some of the conditions of applicability of (2.15) are discussed in Zakharov *et al.* (1992). It is important to note that the asymptotic derivation does not allow correction terms to be added systematically, i.e. to pass to higher approximations. In particular, higher-order nonlinear terms (five-wave processes) have not been considered so far. Meanwhile, it is well-known that taking account of these terms can lead to dynamically important effects, and may be responsible for the formation of long-lived coherent surface patterns, which indicate a deviation from Gaussianity (Annenkov & Shrira 1999).

The fact that (2.15) is obtained as a large-time limit of the theory requires a more detailed discussion. If we drop the assumption that $\mu/\Delta\omega \ll 1$, i.e. include faster variability of statistical moments of the wave field, we, strictly speaking, should use instead of (2.21) the exact solution to (2.20) in the form

$$J^{(1)}_{0123}(t) = -2\mathrm{i}T_{0123}\int_0^t \mathrm{e}^{-\mathrm{i}\Delta\omega(\tau-t)} f_{0123}\,\mathrm{d}\tau + J^{(1)}_{0123}(0)\mathrm{e}^{\mathrm{i}\Delta\omega t}. \tag{2.23}$$

Recently, this point was highlighted by Janssen (2003), who by taking f_{0123} out of the integral in (2.23) and setting $J^{(1)}_{0123}(0)=0$ arrived at a modification of the KE in the form

$$\frac{\partial n_0}{\partial t} = 4\int T^2_{0123} f_{0123}\delta_{0+1-2-3} R_i(\Delta\omega, t)\,\mathrm{d}\boldsymbol{k}_{123}, \tag{2.24}$$

where $\Delta\omega=\omega_0+\omega_1-\omega_2-\omega_3$, and

$$R_i(\Delta\omega, t) = \frac{\sin(\Delta\omega t)}{\Delta\omega}.$$

Then, in the limit $t\to\infty$ (2.24) gives

$$\lim_{t\to\infty} R_i(\Delta\omega, t) = \pi\delta(\Delta\omega),$$

leading to (2.15), while for small times, although still large compared to characteristic wave periods (Janssen 2003),

$$\lim_{t\to 0} R_i(\Delta\omega, t) = t.$$

It is easy to see that while for large times the characteristic time of the evolution is proportional to ε^{-4}, in accordance with the well-known property of (2.15), for small times it is scaled as ε^{-2}.

It is also worth noting that while the classical version of the KE (2.15) creates a false impression that the spectral evolution is caused by exactly resonant interactions only, the modified equation (2.24) via the explicit presence of function $R_i(\Delta\omega, t)$ highlights the importance of non-resonant interactions in two different ways. First, it emphasizes the fact that the classical delta function in (2.15) is a specific large-time limit: for any finite time the width of $R_i(\Delta\omega, t)$ remains finite and, thus, implies the importance of near-resonant interactions for any time; second, the fact that at small times there is a fast spectral evolution entirely due to near-resonant interactions suggests a special significance of the early stages of evolution.

Unfortunately, the range of applicability of equation (2.24) is not clear. Indeed, its derivation is based upon the implicit assumption that $\mu/\Delta\omega \ll 1$. As already mentioned, in generic situations $\Delta\omega$ for four-wave interactions is $O(\varepsilon^2)$, which requires $\mu \ll \varepsilon^2$ and, thus, contradicts the small-time scaling $\mu\sim\varepsilon^2$ implied by (2.24). Hence, in generic situations (2.24) is not self-consistent at the initial stages of evolution, which it aims to describe.

In generic situations there is no small parameter to be utilized, and in order to obtain a kinetic equation valid at all stages of evolution, including the initial one, one should use the closed-form solution (2.23) without any further simplifications. The resulting kinetic equation is

$$\frac{\partial n_0}{\partial t} = -4\mathrm{Re}\int\left\{T^2_{0123}\left[\int_0^t \mathrm{e}^{-\mathrm{i}\Delta\omega(\tau-t)} f_{0123}\,\mathrm{d}\tau\right] + \mathrm{i}T_{0123}J^{(1)}_{0123}(0)\mathrm{e}^{\mathrm{i}\Delta\omega t}\right\}\delta_{0+1-2-3}\,\mathrm{d}\boldsymbol{k}_{123}. \tag{2.25}$$

Thus, in the general setting the evolution of spectral density n depends not only on the initial distribution of n, but also on the initial distribution of $J^{(1)}_{0123}(0)$. A zero value of $J^{(1)}_{0123}(0)$ corresponds to situations where the wave field is initially free, so that the wave components are not correlated, and waves begin to interact only after $t=0$. Thus, this type of initial condition is special. Non-zero $J^{(1)}_{0123}(0)$ corresponds to generic initial conditions where the wave field has been evolving for a sufficiently long time, being governed by the same equations, and small but non-zero correlators

have emerged due to nonlinear interactions. Evidently, the effect of non-zero $J^{(1)}_{0123}(0)$ rapidly decays with time, due to the factor $e^{i\Delta\omega t}$ and integration in $\boldsymbol{k}$-space.

At large times equation (2.25) tends to the classical KE (2.15). For small time-asymptotics, the right-hand side of (2.25) can be expanded in a Taylor series with respect to time, which yields an easily solvable ODE:

$$\frac{\partial n_0}{\partial t} = \alpha + \beta t, \tag{2.26}$$

where α and β are $\sim O({n_0}^3)$ constants. Then, at very short times $t \sim O(\varepsilon^{-1})$ the term βt can be neglected, and the evolution is slow, with time scale $O(\varepsilon^{-4})$. This very initial part of the evolution was considered by Tanaka (2001*a*). At larger but still small times the term βt becomes dominant, which implies evolution with time scale $O(\varepsilon^{-2})$ at $t \simeq O(\varepsilon^{-2})$.

In the next section, we will perform a DNS of the evolution of simple 'toy' wave systems and a quantitative comparison with the corresponding solutions of (2.15), in order to clarify the effects of the approximations involved. For this purpose, we will use the numerical algorithm for the integration of the Zakharov equation described in Annenkov & Shrira (2001). The accuracy of these simulations is controlled by the conservation of the Hamiltonian and other integrals (generally, accuracy of eight significant digits is maintained). However, it is important to note that this control of accuracy refers to dynamical computations. The evolution of statistical characteristics can be verified only by self-consistency, which is best achieved by a detailed study of a simple wave system, which is performed below.

3. The simplest toy model; the role of the exactly resonant, approximately resonant, and non-resonant interactions

3.1. *The simplest toy model*

In order to check the hypotheses used in the derivation of the KE and to address the basic questions on the role of resonant and non-resonant interactions, it is instructive to examine the simplest possible model with a non-trivial evolution of the spectrum. This will also help in developing and testing an adequate DNS method. Here, it is important to note that the method, based on the Zakharov equation, is not restricted to regular grids, so that we are free to consider a wave system with any number of interacting harmonics.

The most elementary model is, clearly, a single resonant quartet

$$\boldsymbol{k}_0 + \boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3, \quad \omega_0 + \omega_1 = \omega_2 + \omega_3; \tag{3.1}$$

an example is shown in figure 1. The kinetic equation for this particular model can be obtained from (2.15) by discretization, or derived directly by using the random phase approximation (Kadomtsev 1982) in the form

$$\frac{\partial n_0}{\partial t} = 8\pi T^2_{0123}[n_2 n_3(n_0 + n_1) - n_0 n_1(n_2 + n_3)]. \tag{3.2}$$

It is worth noting that both sides of (3.2) have the same dimension, due to the presence on the right-hand side of a dimensional constant ('phases correlation time') which is equal to unity in our setting and therefore omitted (cf. Kadomtsev 1982; Rabinovich & Trubetskov 1989). Our choice of the phase correlation constant is based on the requirement that the 'discrete' limit of the usual kinetic equation (2.15) for strongly localized wave packets and the discrete equation for these packets coincide.

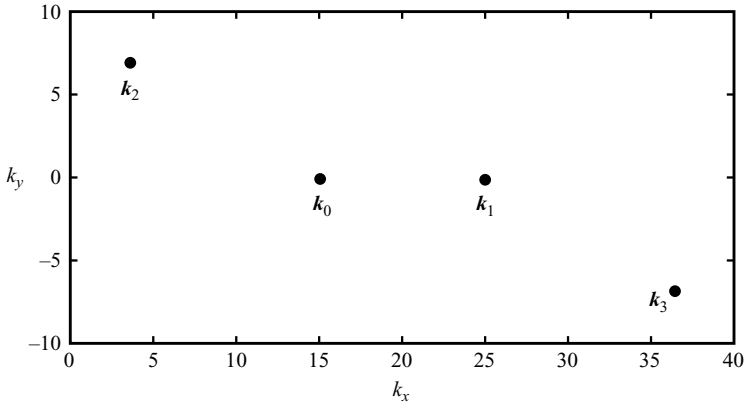

FIGURE 1. Resonant quartet (3.1) in Fourier space. Dots show position of harmonics.

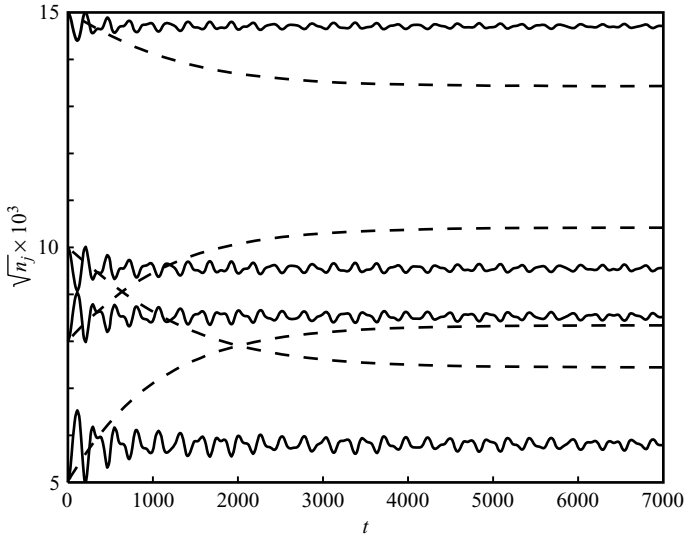

FIGURE 2. Dashed curves: numerical solution of the KE for the single resonant quartet (3.1), for initial conditions (3.3). Solid curves: ensemble-averaged evolution for 10000 quartets (3.4), with $\Delta = 0.6$ and the same initial conditions, obtained by the numerical solution of (2.14). Each realization corresponds to a particular choice of random initial phases and of random mismatch in (3.4).

Numerical solution of (3.2) for quartet (3.1), for initial conditions

$$b_0(0) = 1.5 \times 10^{-2}, \quad b_1(0) = 1.0 \times 10^{-2}, \quad b_2(0) = 0.8 \times 10^{-2}, \quad b_3(0) = 0.5 \times 10^{-2}, \tag{3.3}$$

$n_i(0) = b_i(0)^{1/2}$, is shown in figure 2 (dashed curves). Here and below, we prefer to plot and discuss the evolution of amplitudes $|b(t)|$, instead of wave packet intensities $n(t) = |b(t)|^2$. Owing to the presence of additional integrals relating the amplitudes of interacting packets (Manley–Rowe integrals), this wave system does not tend to the Rayleigh–Jeans spectrum, despite the fact that the system is conservative (no forcing or dissipation terms are included). In order to obtain the solution by DNS,

the most direct way would be, at first glance, to build an ensemble of realizations of the same quartet, each realization having a different set of random initial phases, with subsequent averaging. This was performed by Stiassnie & Shemer (2005), who noted that the result was qualitatively different from the evolution obtained with the kinetic equation. Since, as it was mentioned above, the correct procedure should take into account approximate resonant interactions as well, we, for our first attempt, introduce a frequency mismatch into each realization, so that each quartet has the form

$$\hat{\boldsymbol{k}}_0 + \hat{\boldsymbol{k}}_1 = \hat{\boldsymbol{k}}_2 + \hat{\boldsymbol{k}}_3, \quad \hat{\omega}_0 + \hat{\omega}_1 = \hat{\omega}_2 + \hat{\omega}_3 + \hat{\Omega}, \tag{3.4}$$

where $\hat{\boldsymbol{k}}_i = \boldsymbol{k}_i + \boldsymbol{K}_i$, $\hat{\omega}_i = \omega(\hat{\boldsymbol{k}}_i)$, $\boldsymbol{K}_i$ is a random vector, $|\boldsymbol{K}_i| \leqslant \Delta$, Δ is a parameter and $\hat{\Omega}$ is the frequency mismatch. For each quartet (3.4), the evolution is computed with the Zakharov equation (2.14), with subsequent ensemble averaging. The result is shown in figure 2 (solid curves) and is qualitatively similar to the picture observed by Stiassnie & Shemer (2005). The amplitudes of the harmonics evolve on a short time scale with fast, gradually diminishing, oscillations. Such a scenario of evolution is radically different from the corresponding solution of the kinetic equation: there is not the slightest resemblance. Clearly, an alternative procedure is needed, which (i) would take into account both exactly resonant and approximately resonant interactions, for each realization, and (ii) would not depend on the position of additional harmonics in the Fourier space, or on other parameters.

3.2. *The role of the exactly resonant, approximately resonant, and non-resonant interactions; the idea of clusters*

The most natural way to build a wave system that would satisfy the conditions formulated above, while at the same time corresponding to the case of four resonantly interacting waves, is to consider four interacting *wave packets of finite width* in the Fourier space.

Let us model each packet as a 'cluster' of random-phase harmonics, such that the sum of their amplitudes squared is equal to the total intensity of a packet. In figure 3(*a*), each wave packet is represented as a cluster of five harmonics $\boldsymbol{k}_j$, $\boldsymbol{k}_j \pm \boldsymbol{d}_x$, $\boldsymbol{k}_j \pm \boldsymbol{d}_y$, where $\boldsymbol{d}_x = \Delta\boldsymbol{\kappa}_x$, $\boldsymbol{d}_y = \Delta\boldsymbol{\kappa}_y$, $\boldsymbol{\kappa}_x = (1, 0)$, $\boldsymbol{\kappa}_y = (0, 1)$ are the unit wave vectors. We will refer to the parameter Δ as the *cluster size* in the $\boldsymbol{k}$-space. The procedure of constructing clusters includes a parallel translation of the original resonant quartet by $\boldsymbol{d}_x$, $\boldsymbol{d}_y$, the resulting quartets being in approximate resonance, due to the nonlinear structure of resonance conditions. Simultaneously, in each pair of clusters there are close pairs of harmonics that form approximately resonant quartets. Also, in each cluster a central harmonic interacts with its sidebands, in Benjamin–Feir type interaction. In this way, in the considered 'minimal' configuration, the evolution of a single resonant quartet in the kinetic equation corresponds to the evolution of 181 coupled quartets in the dynamical equation.

Figure 3(*b*) shows the evolution of packet intensities with time obtained by the numerical integration of the Zakharov equation for the wave system specified in figure 3(*a*). Here and below, time is measured in characteristic wave periods. Within each cluster, initial amplitudes of individual harmonics were chosen randomly, but subject to the condition that the total packet intensity is equal to the prescribed initial value, given by (3.3). In this and subsequent simulations, this distribution of amplitudes within each cluster was kept the same for all realizations, and only the random phases of harmonics were different. However, it was verified that if all the amplitudes are random as well, this does not affect the results, provided that the initial conditions for the amplitudes of the clusters are satisfied. The evolution almost

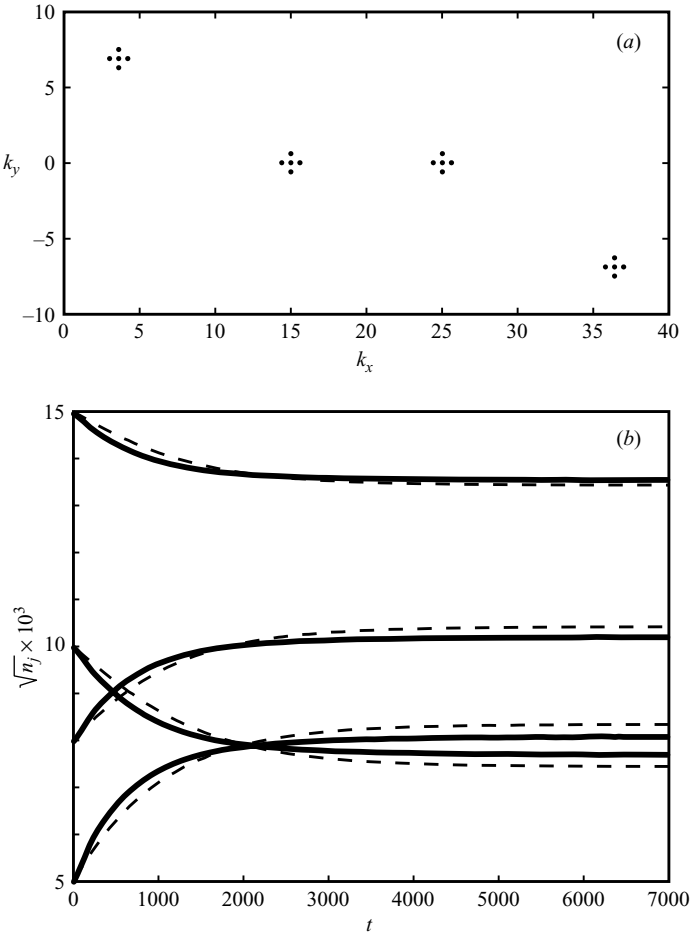

FIGURE 3. (*a*) Resonant quartet (3.1): an example of cluster representation, with the cluster size $\Delta = 0.6$. (*b*) Comparison of the DNS with the KE solution. Solid curves: evolution of amplitudes of four wave packets (3.1) for initial conditions (3.3), averaged over 50000 realizations. Each realization corresponds to a particular choice of random initial phases. Dashed curves: numerical solution of the KE.

coincides with that obtained with the kinetic equation, except at the initial stage where the time scale appears to be slightly different. The problem of time scales will be addressed in the next section; here it is important to note that the numerical solution in figure 3(*b*) is virtually independent of the way the clusters were chosen (i.e. the cluster size Δ, the number and positions of harmonics within the clusters). Specifically, the numerical solution for the example shown in figure 3(*a*, *b*) was found to be independent of Δ in the range $0.05 \leqslant \Delta \leqslant 2.0$. It is however important to choose a configuration of resonances that is most structurally stable, in a certain sense, with respect to the change of amplitude.

3.3. *Other examples of 'discrete models'*

Similar properties are also obtained for much more complex systems consisting of numerous (up to 1000) localized wave packets. As an illustrative (sufficiently rich but still graphically traceable) example, we present the evolution of a system of 24 wave packets. Figure 4 shows the numerical solution for this system: it tends to the

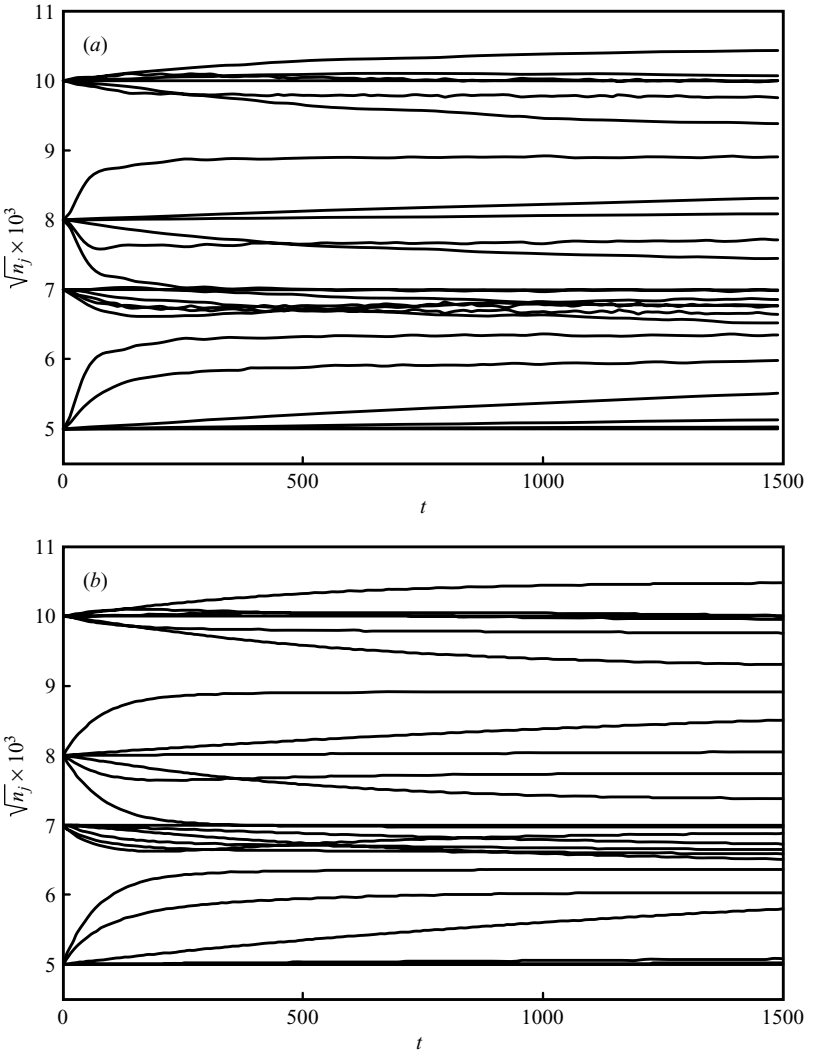

FIGURE 4. Evolution of amplitudes for an ensemble of 24 wave packets. (*a*) DNS, averaged over 6000 realizations, (*b*) solution of the KE.

thermodynamic equilibrium state, and direct numerical simulation beyond the initial stage of evolution virtually coincides with the solution of the kinetic equation.

3.4. *Conclusions: key role of near-resonant interactions and the idea of clusters*

Simulations of a few simple wave systems show that the interactions that are not exactly resonant (i.e. with a small but finite mismatch) play a crucial role in the DNS of the evolution of the statistical properties. An adequate account of both resonant and near-resonant interactions is achieved with a configuration based on the representation of waves by clusters of harmonics. Physically, such a representation corresponds to the interaction of wave packets of finite width in Fourier space. With the exact resonances only, the DNS of a wave system cannot be even qualitatively similar to the numerical solution of the kinetic equation. On the other hand, simultaneously taking into account exact and approximate resonances, even in the most crude way

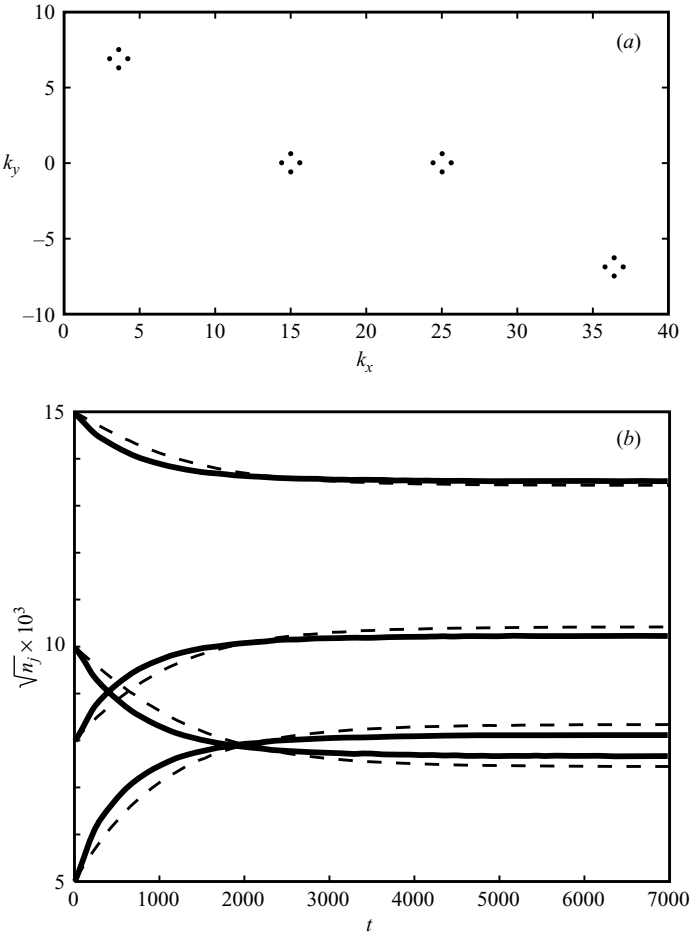

FIGURE 5. (*a*) Cluster representation of the resonant quartet (3.1) with only approximately resonant interactions. (*b*) Solid curves: evolution of amplitudes for this representation, averaged over 50000 realizations. Dashed curves: numerical solution of the KE.

(say, representing a wave packet with just two harmonics) leads to a qualitative similarity, at least. Note that the presence of exact resonances is not essential, i.e. they do not play any special role. To illustrate this, let us slightly modify the clusters shown in figure 3(*a*), by removing the central harmonic from each cluster, so that the resulting wave system contains only approximately resonant interactions (figure 5*a*). Simulation (figure 5*b*), again compared with the same KE solution as before, shows that a wave field can be adequately described by taking into account approximate resonances only. Inclusion of interactions that are far from resonant (with frequency mismatch considerably larger than $O(\varepsilon^2)$) does not have any noticeable effect on the evolution.

The issue of quantitative comparison is more subtle. Before comparing it with other results, the DNS employed should first be validated internally. The extensive testing we carried out showed remarkable robustness of the DNS results obtained by employing different moderately crude representations of wave packets, i.e. models of the clusters. The specific configuration used in the present article has been found to provide the best compromise: an economical way of taking into account the

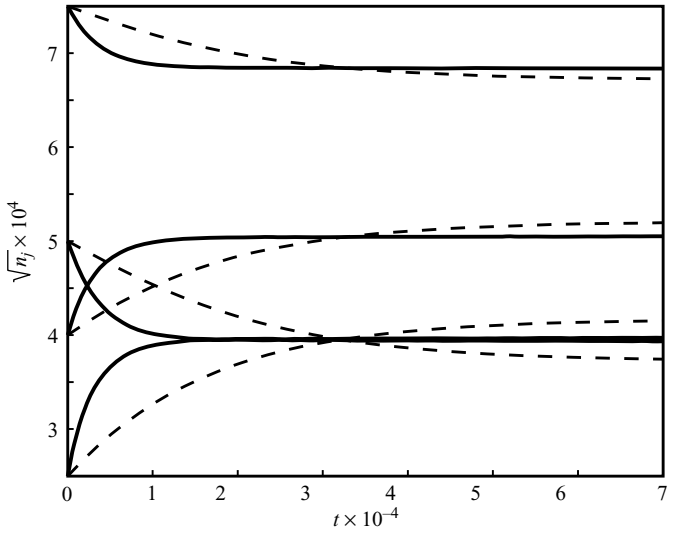

FIGURE 6. As in figure 3(*b*), but for initial amplitudes of all wave packets multiplied by 0.5.

near-resonant interactions without a noticeable dependence on parameters. In this way, we have a relatively simple and robust tool for the quantitative modelling of the statistical evolution of wave ensembles. The tool is used in the next section for a detailed analysis of the wave field evolution and a quantitative comparison of DNS and the KE.

4. Detailed analysis of the wave field evolution

4.1. *The time scales of the field evolution; fast and slow evolution*

In the wave systems chosen for the analysis so far the evolution is quite simple: in the absence of any non-conservative effects the system tends to a certain equilibrium state which depends on the initial conditions. To test with DNS the predictions of field evolution requires, first of all, a careful comparison of the evolution time scales.

As we discussed earlier in § 2.3, it is expected that the amplitude dependence of the 'true' time scales obtained with DNS will differ from the strict ε^{-4} proportionality implied by the classical kinetic equation.

In figure 3, we have plotted the evolution of the toy model obtained with the KE and DNS, for a specific characteristic steepness of the waves, close to 0.1. Although the curves are quite close, a certain discrepancy in the time scales can be seen. Moreover, the time scales of the evolution are found to have different amplitude dependence. Repeating the numerical simulations for half-amplitudes, taking the initial values of b_i as

$$\hat{b}_i(0) = 0.5 b_i(0), \quad i = 0, 1, 2, 3, \tag{4.1}$$

we arrive at the somewhat different picture presented in figure 6. While the KE solution time scale is as always proportional to ε^{-4}, the DNS shows a ε^{-2} dependence. This can be seen in figure 7, where the evolution of amplitudes multiplied by 0.5, 1.0 and 2.0 is plotted versus time normalized by ε^{-2}. Again, no particular dependence on the cluster size was found, provided that $0.05 < \Delta < 1.0$. Thus, for small amplitudes the evolution is much faster according to DNS than within the framework of the KE.

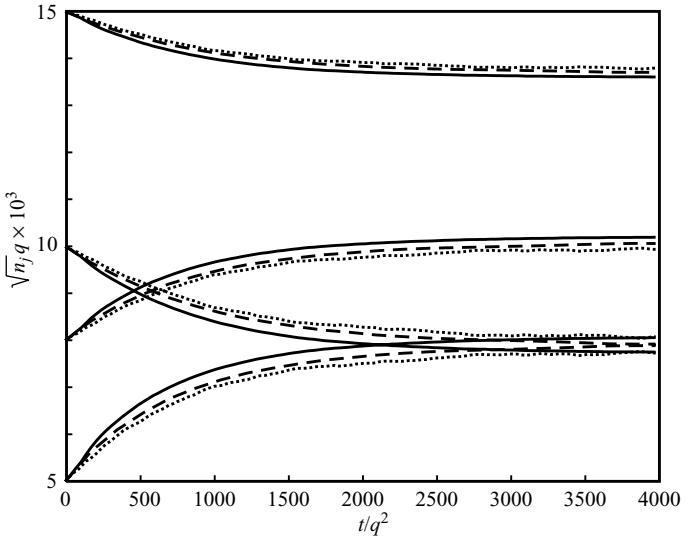

FIGURE 7. Evolution of amplitudes of four wave packets (3.1), normalized by q^{-1}, for $q = 1$ (solid curves), $q = 0.5$ (dashed curves), $q = 2$ (dotted curves), with time scales normalized by q^2. The averaging is over 50000 realizations.

The behaviour demonstrated in this example was found to be typical of all the systems made up of a finite number of localized packets: in any conservative wave system that is initially far from equilibrium, practically all the evolution towards the equilibrium occurs on the 'fast' ε^{-2} time scale, contrary to the KE predictions.

To study evolution on a slow time scale we have to modify the systems under consideration. Let us now add appropriately chosen small non-conservative effects. This can be done by formally replacing each ω_i, where $i = 0, 1, 2, 3$, by $\omega_i + \mathrm{i}\gamma_i$, where γ_i are sufficiently small, in order to retain the Hamiltonian structure at the desired order of approximation. Results of the DNS with the initial conditions (4.1) and

$$\gamma_0 = \gamma_1 = -7 \times 10^{-7}, \quad \gamma_2 = 2.5 \times 10^{-7}, \quad \gamma_3 = 1.705 \times 10^{-7}, \tag{4.2}$$

together with the corresponding KE solution, are shown in figure 8. While the time scale of the initial evolution in the DNS is, as previously, proportional to ε^{-2}, the subsequent 'slow' evolution of the non-conservative system has the same time scale for both the KE and DNS.

4.2. *Evolution of correlators*

Apart from comparing calculations of the evolution of second-order moments obtained within the framework of DNS and KE, the DNS enables us to calculate explicitly the evolution of statistical moments of any order. The issue of the evolution of higher moments is an interesting and completely open question in itself. It is worth noting, however, that because of wave breaking it is not *a priori* clear that any asymptotic procedure employing a truncation in the Hamiltonian can capture accurately the higher moments, and, therefore, since most of the existing DNS approaches are based upon an asymptotic procedure, any conjectures concerned with the true evolution of higher moments should be considered with caution. However, in this paper we are concerned primarily with checking specific hypotheses involved in the derivation of the KE, and the issues arising from the truncations in the Hamiltonian are put aside.

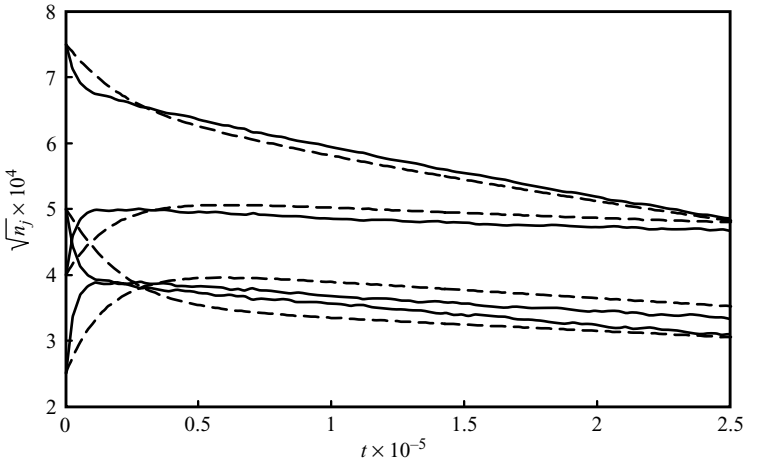

FIGURE 8. Evolution of the resonant quartet for the same initial conditions as in figure 6, with small non-conservative effects added. The averaging is over an ensemble of 1000 realizations.

The hypothesis of quasi-Gaussianity should be justified by the direct calculation of the fourth-order statistical moment. Let us first perform this calculation for the model consisting of four wavepackets that was considered in § 3. Although the number of harmonics in this model are too small for Gaussianity, we can nevertheless verify the crucial assumption that the wave field is close to a free field, i.e. all higher-order correlators approximately decompose into products of pair correlators. Let us now consider equation (2.19), which links the evolution of the second statistical moment for each harmonic with the sum over the fourth-order cumulants, multiplied by the corresponding interaction coefficients, for all the interactions involving this harmonic. The perturbation expansion used in the derivation of the KE is valid provided that the wave field is close to the free field, i.e. this sum of cumulants must be small compared to the corresponding sum of fourth-order correlators. Let us calculate the right-hand side of (2.19) for all harmonics comprising the first cluster (corresponding to wavepacket $\boldsymbol{k}_0$ in figure 1), and compare it to the zeroth-order value, summing over the real parts of fourth-order correlators. In figure 9, the function

$$G(t) = \mathrm{Im} \sum_{j=1}^{5} \sum T_{jklm} \langle b_j^* b_k^* b_l b_m \rangle \tag{4.3}$$

is plotted, where the first summation refers to all the harmonics comprising the cluster, and the second one includes all the interactions for the jth harmonic. Note that the time integration of $2G(t)$ gives the evolution of n_0 shown in figure 3. For comparison, in the same figure we plot the function

$$G^{(0)}(t) = \sum_{j=1}^{5} \sum T_{jklm} J_{jklm}^{(0)}, \tag{4.4}$$

replacing the correlators by their decomposition into the corresponding products of second-order correlators. As it is seen from figure 9, the first-order correction is three orders of magnitude smaller than the corresponding free-field value. Hence, the key assumption of the theory, that the wave field remains quasi-Gaussian in the process of evolution, is well justified.

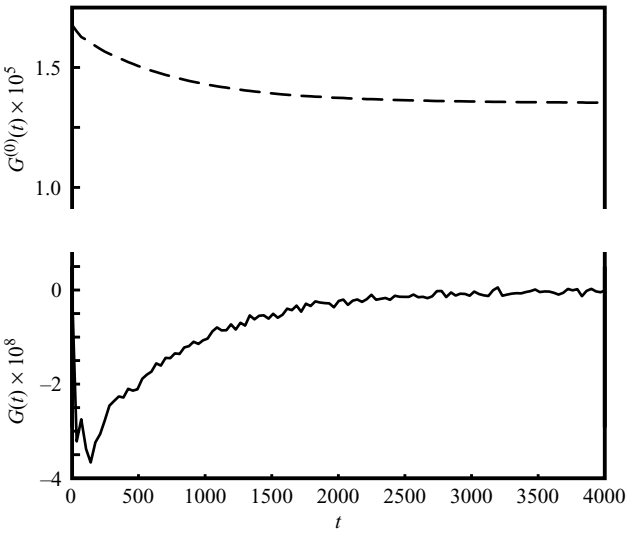

FIGURE 9. Solid curve: evolution of $G(t)$ for the first cluster (corresponding to wavepacket $\boldsymbol{k}_0$ in figure 1). Dashed curve: evolution of the corresponding free-field function $G^{(0)}(t)$.

For a wave field consisting of a larger number of harmonics, the deviations from Gaussianity can be checked by tracing the evolution of kurtosis, related to the fourth moment $\langle\zeta^4\rangle$, where

$$\langle\zeta^4\rangle = \frac{1}{4}\int (\omega_1\omega_2\omega_3\omega_4)^{1/2}\langle b_1^* b_2^* b_3 b_4\rangle \,\mathrm{d}\boldsymbol{k}_{1234} + \text{c.c.}, \tag{4.5}$$

and the kurtosis $\mathscr{K}$ is

$$\mathscr{K} = \frac{\langle\zeta^4\rangle}{\langle\zeta^2\rangle} - 3. \tag{4.6}$$

With the increase of the number of harmonics with random phases, the wave field approaches Gaussianity, and the kurtosis tends to zero from below. For a wave field consisting of 24 packets, the evolution of the kurtosis is shown in figure 10. This wave field was discussed in § 3.3, and the evolution of amplitudes was shown in figure 4. Figure 10 shows that the kurtosis remains small during the evolution, so that the wave field remains close to Gaussianity. It should be emphasized that these simulations of the kurtosis are of qualitative character: the chosen number of harmonics is adequate for capturing the evolution of the second moments but is not sufficient for the more sensitive kurtosis. The kurtosis is expected to be zero at the initial moment for any sufficiently large ensemble of non-interacting waves, and remain small and positive in the course of evolution because of the specifics of the deep water interaction coefficients (Janssen 2003). Although in this simulation the kurtosis is negative due to the insufficient number of harmonics, it remains close to its initial value and, in accordance with expectations, slightly exceeds it for most of the evolution.

4.3. *Conclusions: phenomenon of 'fast' evolution due to near-resonant interactions*

The discussion of the derivation of the KE in § 2 suggests that the true time scales of evolution must be different from the KE predictions at a certain initial stage of evolution. This initial stage of the evolution has time scale proportional to ε^{-2}, and it is implicitly assumed that the main stage of evolution occurs on an ε^{-4} time scale. So one could expect to find in DNS a faster initial evolution followed by an ε^{-4}

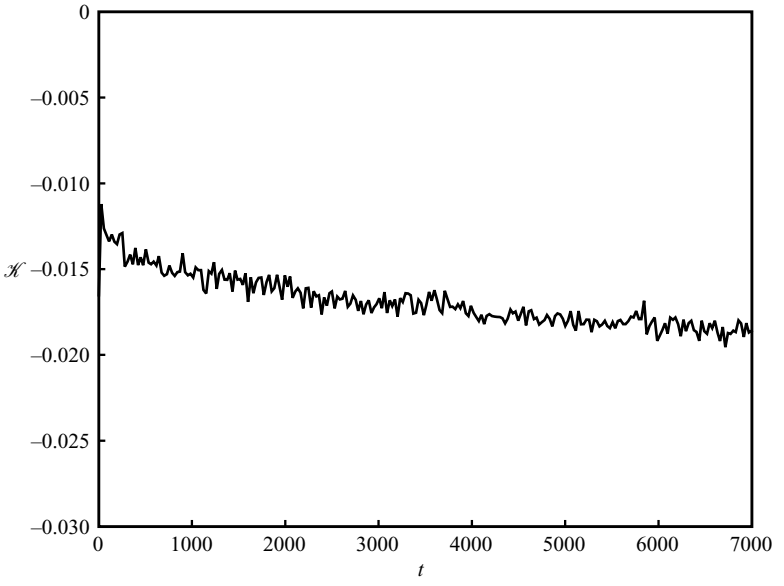

FIGURE 10. Evolution of kurtosis for the wave field consisting of 24 packets, with the averaging over 6000 realizations.

evolution governed by the KE. However, we found that if a conservative wave system is initially far from statistical equilibrium, all of its evolution towards this equilibrium occurs on the fast time scale. On the other hand, if small non-conservative effects are included, the subsequent slow evolution is correctly predicted by the KE.

This important fact has two consequences. First, the fast adjustment towards equilibrium is not resolved by the KE and, in fact, represents a modification of its initial conditions. Second, the reaction to any rapidly varying forcing would be incorrectly treated by the KE, since the system will evolve to equilibrium with a different time scale. An example of such a scenario is presented in figure 11.

Moreover, since the 'dynamical' wave field reacts to a fast external forcing on a different time scale than the 'statistical' one, this reaction can be different. A smaller reaction time scale means that the wave field can have sufficient time to adapt to an external forcing, which is instantaneous in terms of the KE. The consequences of this fact for more complex wave systems are yet to be elaborated.

It should be noted that all the above simulations were carried out for situations corresponding to 'cold start', i.e. $J^{(1)}_{0123} = 0$ at $t = 0$. Therefore, a legitimate question is whether the conclusion on the evolution time scale holds for generic situations with non-zero initial $J^{(1)}_{0123}$, caused by the preceding evolution of the wave field. However, since all the evolution in figure 7 obeys ε^{-2} scaling, any $t = t^*$ can be chosen as the initial moment, without affecting the conclusion. The most convincing argument is provided by figure 11, which shows the presence of fast evolution in a system that experiences an abrupt variation of forcing after $O(\varepsilon^{-4})$ wave periods, that is with definitely non-zero correlations between the harmonics by this moment.

5. Effect of quintet interactions

The problem of the role of higher-order nonlinearities in the evolution of random wave fields is a major open question. It is well-known that these processes could be

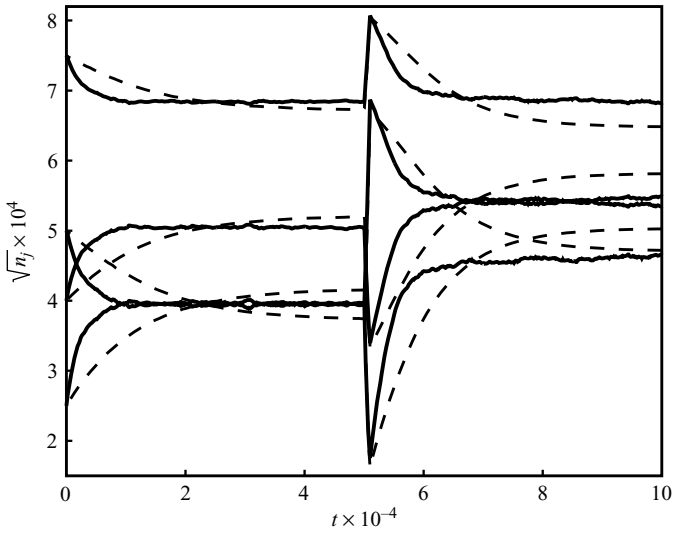

FIGURE 11. Evolution of the resonant quartet for the same initial conditions as in figure 6, with rapidly varying forcing. After relaxation of the conservative wave system to equilibrium, non-conservative effects are inserted at $t = 5 \times 10^4$, with $\gamma_0 = 1 \times 10^{-4}$, $\gamma_1 = 2.5 \times 10^{-4}$, $\gamma_2 = \gamma_3 = -3.5 \times 10^{-4}$, and then switched off at $t = 5.1 \times 10^4$. The averaging is over 1000 realizations.

dynamically important, and, in particular, are responsible for the formation of long-lived horseshoe coherent patterns (Annenkov & Shrira 1999). Watson & Buchsbaum (1996) and Watson (1999) have demonstrated the importance of higher-order (in their case, four-wave) interaction processes for the evolution of capillary wave spectra. Also, there are grounds to believe that higher-order interactions are even more important for the evolution of a random gravity wave field. Indeed, the characteristic time scale for four-wave processes in the dynamical equations is the same as the time scale of spectrum evolution within the three-wave Hasselmann equation, while the dynamical time scale due to quintet interactions ($O(\varepsilon^{-3})$) is faster than the $O(\varepsilon^{-4})$) characteristic time scale typical of the evolution of the gravity wave spectrum. Moreover, there exist phase-locked quintet interactions, which often manifest themselves in the emergence of crescent shaped wave patterns in natural conditions (Csanady 1984). These phase-locked quintet interactions are expected to retain their $O(\varepsilon^{-3})$ time scale even in the random field setting; therefore, the question of their role seems to be intriguing. Since so far no asymptotic statistical theory accounting for high-order nonlinear processes (five-wave interactions) has been developed, the DNS seems to be the only way forward. In this section, as a first step, we will examine the evolution of the same wave systems consisting of a finite number of spectrally narrow wave packets considered in the previous sections, but taking into account five-wave interactions.

Let us first consider the simplest 'pure' five-wave model: a single resonant quintet without four-wave processes

$$3\boldsymbol{k}_0 = \boldsymbol{k}_1 + \boldsymbol{k}_2, \quad 3\omega_0 = \omega_1 + \omega_2 + \Omega. \tag{5.1}$$

Because the five-wave resonance domain is narrow ($O(\varepsilon^3)$), linear resonances are of little significance since the nonlinear frequency shift caused by quartet interactions far exceeds the width of this domain. Thus, in (5.1) we have introduced explicitly

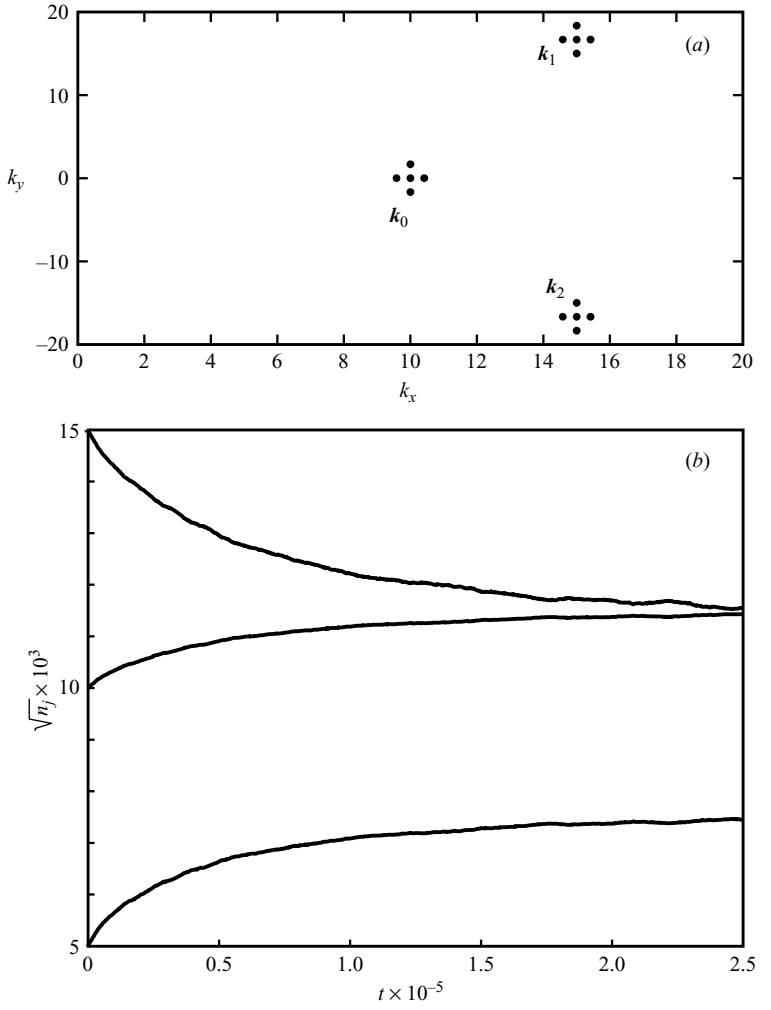

FIGURE 12. (*a*) Three wave packets forming resonant quintet (5.1) in Fourier space. (*b*) Evolution of quintet (5.1) for initial conditions (5.2), with the averaging over 1000 realizations.

a mismatch Ω, taking into account the nonlinear frequency correction and ensuring that wave packets of finite amplitude take part in a strong five-wave interaction.

The quintet (5.1) is shown in figure 12(*a*). Results of the DNS, for the initial conditions

$$b_0(0) = 1.5 \times 10^{-2}, \quad b_1(0) = 0.5 \times 10^{-2}, \quad b_2(0) = 1.0 \times 10^{-2} \tag{5.2}$$

and $\Omega = 0.0161$ are shown in figure 12(*b*). Here, we use clusters of the same form as in § 3, with $\Delta = 0.6$, and the choice of the cluster parameters is again found to be of no importance.

This simulation demonstrates that five-wave processes play a significant role in the evolution of spectra, even for wave packets of relatively small intensities. Note that in the simulation shown in figure 12, the initial steepness of interacting waves does not exceed 0.1. The time scales of the evolution are considerably larger than the characteristic time scales of four-wave interactions.

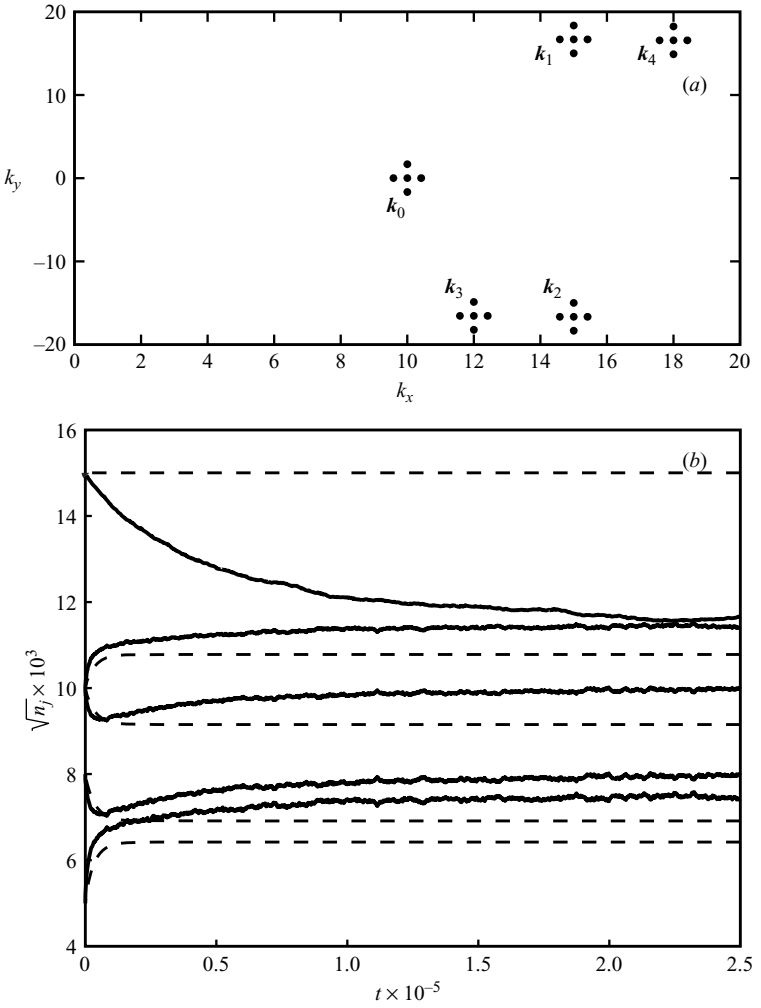

FIGURE 13. (*a*) 'Mixed' wave system (5.3) consisting of five wave packets, linked by simultaneous four-wave and five-wave interactions. (*b*) Evolution of 'mixed' wave system (5.3), with averaging over 1000 realizations. Solid curves: DNS for initial conditions $b_0(0) = 3.75 \times 10^{-2}$, $b_1(0) = 2.5 \times 10^{-2}$, $b_2(0) = 2.0 \times 10^{-2}$, $b_3(0) = 1.25 \times 10^{-2}$, $b_4(0) = 2.5 \times 10^{-2}$, and $\Omega_1 = 0.084$, $\Omega_2 = 0.085$. Dashed curves: Solutions of the KE (without five-wave interactions).

The most important question is whether five-wave processes can be significant at the four-wave interaction time scale. Let us consider a 'mixed' wave system with wave packets taking part simultaneously in four- and five-wave processes in the form

$$3\boldsymbol{k}_0 = \boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4, \quad 3\omega_0 = \omega_1 + \omega_2 + \Omega_1 = \omega_3 + \omega_4 + \Omega_2. \tag{5.3}$$

This wave system, shown in figure 13(*a*), contains two resonant quintets (with frequency mismatches Ω_1, Ω_2 chosen to ensure the maximal five-wave modulations) and one approximately resonant quartet. For this simulation, the characteristic steepness was taken to be rather large, of the order of 0.25.

Results of the DNS, shown in figure 13(b), demonstrate that for large, although not unrealistic, steepness of the interacting waves the five-wave processes could significantly affect the field evolution, even on the time scale of four-wave interactions.

6. Concluding remarks

We begin by summarizing the main findings. First, we demonstrated the key importance of near-resonant interactions for the nonlinear evolution of the statistical characteristics of wave fields. Although the well-established kinetic equation of the existing statistical theory includes only exactly resonant interactions, we emphasize that any DNS must take into account approximately resonant interactions, and have demonstrated that the inclusion of only exact resonances gives physically meaningless results, while the presence or absence of exact resonances in the system is not important. Two points are new here and should be highlighted. First, the evolution of a wave system is relatively insensitive to the specific procedure of taking into account the near-resonant interactions. This justifies the use of clusters and leads to an efficient way of performing DNS. The second point is concerned with the phenomenon of 'fast' (with time scale $O(\varepsilon^{-2})$) spectral evolution caused entirely by near-resonant interactions. The existence of such fast evolution was expected to occur at the $O(\varepsilon^{-2})$ initial stage; this stage, perceived to be a mere 'transient' to normal ε^{-4} evolution, is usually ignored. However, if a conservative wave system is initially far from its statistical equilibrium, then all or most of its evolution towards this equilibrium was found to occur on the fast time scale. When a wave system does exhibit a slow ε^{-4} time scale evolution, say, in the presence of appropriate forcing, it goes in quantitative accordance with the prediction of the KE.

The effect of higher-order interactions on the evolution of wave spectra (for any wave field) was investigated, also for the first time. Even for small steepnesses quintet interactions can significantly affect the spectrum evolution, although on a much slower time scale. However, for waves of high, but still not unrealistic, steepness $\varepsilon \simeq 0.25$ the scales of evolution become no longer separable.

By tracing the evolution of high statistical moments of the wave field, we directly verified one of the main assumptions used in the derivation of the kinetic equation: quasi-Gaussianity of the wave field holds throughout the evolution, both with and without taking account of quintet interactions.

Now let us briefly outline some of the implications and open questions. The acquired understanding of the role of near-resonant interactions enables us to develop an efficient DNS approach based upon integration of the Zakharov equation, the use of which is in no way confined to the simple models considered in this paper.

The conclusions regarding the key importance of the fast evolution will remain valid for the more common situation of a generic continuous wave field (i.e. without the localized packet structure, see Annenkov & Shrira 2006). There are grounds to expect the importance of fast evolution not only at the initial stage (which could be understood as an effective modification of the initial conditions in the classical KE), but also in the presence of rapidly varying forcing (for example, wind gusts). In the context of wind waves, our results suggest that the modified version of the kinetic equation, (2.24) derived by Janssen (2003) and (2.25) suggested in this paper, could be a noticeably better tool for description of the field evolution compared to the classical KE (2.15). Whether the use of a better but more computationally expensive tool (the integration is performed on the ε^{-2} time scale compared to ε^{-4}) is justified, should probably be decided for each particular situation individually. No doubt, the issue

requires special investigation. It should be mentioned, however, that at the moment there is no numerical code for integrating the modified kinetic equations, and it took almost twenty years to develop reasonably efficient codes for the classical KE.

We would also like to point out the class of physical situations where the simple field configurations we considered represent a reasonably adequate description of reality. For example, the evolution of paddle waves in wave tanks (especially with oblique waves with wall reflections taken into account) certainly falls into this class, although in this context the wave field evolution is rarely viewed from the perspective of spectrum (or other statistical properties) evolution. Most of the spectral evolution in these circumstances must occur at the fast stage. In particular, the spectral evolution observed in the experiments by Shemer *et al.* (2001) was entirely due to near-resonant interactions (since in the one-dimensional setting the exact ones are prohibited).

Throughout the paper, we considered gravity waves on the surface of a deep fluid, for which the dispersion relation does not allow triad interactions to occur. However, it is well-known that in shallow finite-depth water three-wave interactions can become approximately resonant, although exact triad resonances are still prohibited. Results of this paper suggest that these approximate resonances can play an important role in the spectral evolution.

The most physically interesting situations are concerned with continuous wave fields maintaining the spectral cascades of wave turbulence. By employing the DNS method based upon the findings of this paper we can address both a number of fundamental questions concerned with evolution of statistical properties of random wave fields, and numerous problems of applied character that are specific for wave fields of different nature (recall that our results are of general character and not confined to water waves). The first results for continuous wave fields with spectral cascades are reported in Annenkov & Shrira (2006).

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